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Group-Theoretic Analysis of the Mixing Angle in the Electroweak Gauge Group

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In this paper we provide strong mathematical support for the idea that the experimentally measured magnitude $1 - M_W^2/M_Z^2$ associated with $\sin^2 \theta_W$ in the standard model of electroweak interactions cannot be simultaneously identified with the squared quotient of the electric charge by the $SU(2)$ charge, e^2/g^2. In fact, the natural, mathematical requirement that the Weinberg rotation between the gauge fields associated with the third component of the "weak isospin" (T_3) and the hypercharge (Y) proceeds from a *global Lie-group homomorphism* of the $SU(2) \otimes U(1)_Y$ gauge group in some locally isomorphic Lie group [which then proves to be $U(2)$], and not from a *local* (Lie algebra) *isomorphism*, imposes strong restrictions so as to fix the single value $e^2/g^2 = 1/2$. The two definitions of $\sin^2 \theta_W$ can only be identified in the asymptotic limit corresponding to an earlier stage of the universe before the spontaneous symmetry breaking had taken place.

1. INTRODUCTION

There are two basic ingredients in the constitution of a model to describe the unified electroweak interactions, the Weinberg–Salam–Glashow standard model, which deserve further study and which lessen the (mathematical) beauty of the theory as a whole. One is the way in which the W-Z-bosons acquire mass, the Higgs mechanism, and the other is the rotation between the gauge fields A^3_μ and A^4_μ associated with the third component of weak isospin T_3 and the hypercharge Y,

$$Z^0_\mu = \cos \theta_W A^3_\mu - \sin\theta_W A^4_\mu \qquad (1)$$
$$A_\mu = \sin \theta_W A^3_\mu + \cos \theta_W A^4_\mu$$

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intended to define the proper electromagnetic field, with a rather imprecise connection to the "weak" Gell'Mann–Nishijima relation

$$Q = T_3 + \tfrac{1}{2}Y \tag{2}$$

meant to define a proper electric charge in the Lie algebra. Furthermore, this rotation is presented in the literature in two conceptually different ways. In the first approach, the most conventional one, it is performed on the vector potentials in order to diagonalize the mass matrix resulting in the Lagrangian after the Higgs mechanism is applied. The corresponding rotation angle is directly related to a quotient between masses of intermediate vector bosons. This approach thus combines the two above-mentioned negative ingredients of the theory.

However, there is a second approach which is previous to any mechanism intended to supply mass to the vectorial bosons of the theory. Now the transformation (1) is a consequence of (dual to) a transformation on the Lie algebra generators involving T_3 and Y, providing an electric charge through relation (2) and its counterpart which defines the neutral weak charge associated with Z^0_μ. It must be a rotation in order to keep the canonical independence of the gauge fields (Bernstein, 1974). The corresponding angle is directly related to the quotient between new and old coupling constants e^2/g^2.

The conceptual difference between the two characterizations of the mixing angle, as a quotient of masses and a quotient of charges, was first pointed out by Passarino and Veltman (1990). In the present paper we shall prove that there are strong mathematical restrictions on the quotient e^2/g^2 so as to fix the single value 1/2, which demonstrates that this magnitude cannot be directly related to the quantity $1 - M_W^2/M_Z^2$ considered in the standard model as a parameter which has to be determined by experiment.

From now on we shall follow the second approach to the characterization of the mixing angle, according to which $\sin^2 \theta_W$ will appear as a function of coupling constants.

As a preliminary comment to motivate the following formal presentation, we stress that the mere embedding of the electromagnetic subgroup $U(1)_Q$ in the torus $T^2 = U(1)_{T_3} \otimes U(1)_Y$, as suggested by (2), imposes nontrivial restrictions (rational values) on the tangent of the *closed* geodesics associated with its generator. Then the additional requirement that $U(1)_{T_3}$ be a subgroup of $SU(2)$ will impose further severe restrictions.

2. GENERAL ANALYSIS

Let us start by exploring the restrictions that appear on the mixing angle as a consequence of the natural consistency requirement that the rotation in the gauge fields (1) comes from an *exponentiable* automorphism on the Lie algebra of $SU(2) \otimes U(1)_Y$. Since the gauge group is not simply connected, it is not true that any automorphism of the Lie algebra can be realized as the derivative of a global group homomorphism (see, e.g., Chevalley, 1946), or, in other words, a differentiable mapping between two locally isomorphic groups providing a given automorphism of the (common) Lie algebra can in general destroy the global group law. Thus it is the specific topologic structure of the starting group $SU(2) \otimes U(1)_Y$ which restricts the number of globally exponentiable transformations in the corresponding Lie algebra.

Disregarding the global structure of the gauge group could affect some important aspects of the theory, such as (a) the existence or not of monopoles and solitons (see, e.g., Arafune *et al.*, 1975; Hon-Mo and Tsun, 1993), (b) the topological properties of the symmetry breaking (see, e.g., Isham, 1981), or (c) the Bohm–Aharonov effect itself (see, e.g., Aharonov and Bohm, 1959), in short, aspects which have something to do with the topology of the gauge group.

To analyze the set of global homomorphisms from $SU(2) \otimes U(1)_Y$ to locally isomorphic groups we can proceed in two different ways: either we study the set of discrete normal subgroups of $SU(2) \otimes U(1)_Y$, which are the possible kernels of those homomorphisms, or we start from the explicit group law of $SU(2) \otimes U(1)_Y$, write the expression of all homomorphisms involving only the toral subgroup $T^2 = U(1)_{T_3} \times U(1)_Y$, and analyze the conditions under which the group law of the entire group is not destroyed. We shall follow the second approach, although we add some comments on the first one at the end.

Let us parametrize the group $SU(2)$ in a coordinate system adapted to the Hopf fibration $SU(2) \to S^2$, the sphere S^2 being parametrized by stereographic projection. The $SU(2) \otimes U(1)_Y$ group law in the local chart at the identity, which nevertheless keeps the global character of the toral subgroup, is

$$\begin{aligned}
\eta'' &= \frac{z_1''}{|z_1''|} = \frac{\eta'\eta - \eta'\eta^* C' C^*}{[(1 - \eta^{*2} C' C^*)(1 - \eta^2 C C^{*\prime})]^{1/2}} \\
C'' &= \frac{z_2''}{z_1''} = \frac{C\eta^2 + C'}{\eta^2 - C'C^*} \qquad (3) \\
C^{*\prime\prime} &= \frac{z_2''^*}{z_1''^*} = \frac{C^*\eta^{-2} + C^{*\prime}}{\eta^{-2} - C^{*\prime} C} \\
\zeta'' &= \zeta'\zeta
\end{aligned}$$

where $\eta \in U(1)_{T_3} \subset SU(2)$, $\zeta \in U(1)_Y$, $C \in \mathbf{C}$, and z_1, z_2 characterize an $SU(2)$ matrix

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$$

The commutation relations between the (right) generators $T_+ \equiv X_{C^*}$, $T_- \equiv X_C$, $T_3 \equiv X_\eta$, and $Y \equiv X_\zeta$ are

$$
\begin{aligned}
[T_3, T_\pm] &= \pm 2T_\pm \\
[T_+, T_-] &= T_3 \\
[Y, \text{all}] &= 0
\end{aligned}
\tag{4}
$$

We shall consider transformations F: $SU(2) \otimes U(1)_Y \to G$, where G is a group locally isomorphic to $SU(2) \otimes U(1)_Y$ to be determined, induced by all homomorphisms of the torus:

$$
\begin{aligned}
\tilde{\eta} &= \eta^{p}\zeta^{p'} \\
\tilde{\zeta} &= \eta^{q}\zeta^{q'} \\
\tilde{C} &= C, \qquad \tilde{C}^* = C^*
\end{aligned}
\tag{5}
$$

where the parameters p, p', q, q' have to be integers for the univaluedness requirement [although these transformations exhaust all possibilities of obtaining (1), it can be proven that they really exhaust all possible homomorphisms (which locally are isomorphisms) on $SU(2) \otimes U(1)_Y$].

After we apply this transformation the group law becomes

$$
\begin{aligned}
\tilde{\eta}'' &= \left(\frac{(\tilde{\eta}')^{1/p}\tilde{\eta}^{1/p} - (\tilde{\eta}')^{1/p}\tilde{\eta}^{-(d+2qp')/dp}\tilde{\zeta}^{2p'/d}\tilde{C}^*\tilde{C}'}{[(1 - \tilde{\eta}^{-2q'/d}\tilde{\zeta}^{2p'/d}\tilde{C}'\tilde{C}^*)(1 - \tilde{\eta}^{2q'/d}\tilde{\zeta}^{-2p'/d}\tilde{C}\tilde{C}^{*\prime})]^{1/2}}\right)^p \\
\tilde{C}'' &= \frac{\tilde{C}\tilde{\eta}^{2q'/d}\tilde{\zeta}^{-2p'/d} + \tilde{C}'}{\tilde{\eta}^{2q'/d}\tilde{\zeta}^{-2p'/d} - \tilde{C}'\tilde{C}^*} \\
\tilde{C}^{*\prime\prime} &= \frac{\tilde{C}^*\tilde{\eta}^{-2q'/d}\tilde{\zeta}^{2p'd} + \tilde{C}^{*\prime}}{\tilde{\eta}^{-2q'/d}\tilde{\zeta}^{2p'/d} - \tilde{C}^{*\prime}\tilde{C}} \\
\tilde{\zeta}'' &= \tilde{\zeta}'\tilde{\zeta}(\tilde{\eta}''\tilde{\eta}'^{-1}\tilde{\eta}^{-1})^{q/p}
\end{aligned}
\tag{6}
$$

where d is the determinant of the matrix

$$
\begin{pmatrix} p & p' \\ q & q' \end{pmatrix},
$$

and this group law is well-behaved (univalued) if

$$
\frac{2p'}{d} = m, \qquad \frac{2q'}{d} = n, \qquad \frac{q}{p} = k, \qquad m, n, k \in Z \tag{7}
$$

which, in particular, implies $p = \pm 1, \pm 2$. These particular values simply state the well-known fact that the only invariant subgroups of $SU(2)$ itself are I (the identity) and Z_2, respectively, the last one corresponding to the standard homomorphism $SU(2) \to SO(3)$.

The commutation relations between the new generators (with a definition analogous to that given above),

$$
\begin{aligned}
[\tilde{T}_3, \tilde{T}_\pm] &= \pm\frac{2q'}{d}\tilde{T}_\pm \\
[\tilde{Y}, \tilde{T}_\pm] &= \pm\frac{-2p'}{d}\tilde{T}_\pm \\
[\tilde{T}_+, \tilde{T}_-] &= p\tilde{T}_3 + q\tilde{Y}
\end{aligned} \tag{8}
$$

can be obtained directly from (6) or by applying the tangent mapping to F, (5), to the old ones. This transformation gives

$$
\begin{aligned}
\tilde{T}_3 &= \frac{q'}{d}T_3 - \frac{q}{d}Y \\
\tilde{Y} &= \frac{-p'}{d}T_3 + \frac{p}{d}Y
\end{aligned} \tag{9}
$$

and provides a generalized Gell'Mann–Nishijima relation and its counterpart, which now appear quantized.

Let us now examine the transformation induced by (5) in the (third – fourth internal components of the) gauge fields. It is given by

$$
\begin{pmatrix} \tilde{A}^3_\mu \\ \tilde{A}^4_\mu \end{pmatrix} = \begin{pmatrix} 1/\tilde{r} & 0 \\ 0 & 1/\tilde{r}' \end{pmatrix}\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r' \end{pmatrix}\begin{pmatrix} A^3_\mu \\ A^4_\mu \end{pmatrix} \tag{10}
$$

where r, r' are the original coupling constants associated with weak isospin and hypercharge, respectively, and $\tilde{r}$, $\tilde{r}'$ are the final ones. In fact, the covariant derivative $D_\mu = \partial_\mu - ig^k_i T_k A^i_\mu$, where i, k run over 1, 2, 3, 4 ($T_4 \equiv Y$), goes to $\tilde{D}_\mu = \partial_\mu - i\tilde{g}^k_i \tilde{T}_k \tilde{A}^i_\mu = D_\mu$. Therefore,

$$
\tilde{A}^l_\mu = (\tilde{g}^{-1})^l_j a^j_k g^k_i A^i_\mu \tag{11}
$$

where a^j_k is the transformation matrix changing coordinates in the Lie algebra [dual to (9)], which contains the central matrix in (10) as a box, and $\mathbf{g} = \mathrm{diag}(r, r, r, r')$ and $\tilde{\mathbf{g}} = \mathrm{diag}(\tilde{r}, \tilde{r}, \tilde{r}, \tilde{r}')$ are the initial and final (bare) coupling constant matrices.

We now impose the requirement that the complete transformation (10) be the Weinberg rotation (1) ($Z^0_\mu \equiv \tilde{A}^3_\mu$, $A_\mu \equiv \tilde{A}^4_\mu$). This results in

$$
\frac{\tilde{r}^2}{\tilde{r}'^2} = -\frac{pp'}{qq'}, \qquad \frac{r^2}{r'^2} = -\frac{q'p'}{qp}, \qquad \tan^2\theta_W = -\frac{qp'}{pq'}, \qquad \tilde{r} = \frac{p}{\cos\theta_W}r \tag{12}
$$

which contain a further restriction: the product of the four integers $pp'qq' < 0$, a condition afterward necessary to have a (true) rotation. Unfortunately, only the rotations $\theta_W = 0, \pi/2$ correspond to automorphisms of the torus ($d = \pm 1$, i.e., without kernel) but they result only in either the identity or interchange of gauge fields. Therefore, nontrivial rotations require $|d| \neq 1$, which means that the transformation F must be a homomorphism having a nontrivial, discrete kernel $H = \ker F$, going from $SU(2) \otimes U(1)_Y$ onto $SU(2) \otimes U(1)_Y/H$. Adding (12) to (7), we arrive at the result

$$\{p = \pm 1 \text{ and } (p' = -kq', k = \pm 1)\} \Rightarrow \{\tan^2 \theta_W = 1, d = \pm 2q'\} \tag{13}$$

For these values of p, p', q, q' the kernel of the homomorphism [see the transformation (5)] is the normal subgroup $H \equiv H_d$:

$$H_d \equiv \Big\{(C, C^*, \eta; \zeta) = (0, 0, 1; e^{i(2s/d)2\pi}), \quad (0, 0, -1; e^{i[(2s+1)/d]2\pi})/s = 0, 1, \ldots, \frac{|d|}{2} - 1\Big\} \tag{14}$$

which is isomorphic, as a group, to $Z_{|d|}$. All these homomorphisms lead to the same value for $\tan^2 \theta_W$ ($= 1$) and, in fact, all can be written as

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} \pm 1 & -kq' \\ \pm k & q' \end{pmatrix} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & q' \end{pmatrix} \tag{15}$$

where the second factor has determinant $\pm q'$ and represents a transformation from $SU(2) \otimes U(1)_Y$ to $SU(2) \otimes (U(1)_Y/Z_{|q'|})$, and the first one has determinant 2 and would take $SU(2) \otimes U(1)_Y$ to $(SU(2) \otimes U(1)_Y)/H_2 \approx U(2)$ by itself. The second factor affects the quotient between the original coupling constants (not the final one), as can be seen in (12), and the generalized Gell'Mann–Nishijima relation (9). Among the possible values for q', only $q' = \pm 1$ provides us with a proper electric charge; the choice of the signs of p, q, p' is a matter of convention and will define either $\tilde{T}_3$ or $\tilde{Y}$ as $\pm$ the electric charge Q. The corresponding homomorphism has Ker $H_2 = \{(0, 0, 1; 1), (0, 0, -1; -1)\}$ and $U(2)$ as the image group (*true gauge group*) (Isham, 1981; O'Raifeartaigh, 1986; LaChapelle, 1994).

3. DISCUSSIONS

Summarizing, the only global homomorphism (except for a trivial sign ambiguity) from $SU(2) \otimes U_Y(1)$ to a locally isomorphic group defining a proper rotation on the gauge fields compatible with the Gell'Mann–Nishijima relation (thus providing an electric charge) is the homomorphism $SU(2) \otimes U_Y(1) \to U(2)$, which leads to the value $\tan^2 \theta_W = 1$ for the mixing angle.

With the usual choice of multiplets in the Lagrangian of the standard model (see, e.g., Ryder, 1985) T_3 and Y have the expressions

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{16}$$

which agree with the usual expressions if the $U(1)$ subgroups are trivially reparametrized by $\alpha = -2i \ln \eta$, $\beta = i \ln \zeta$ ($T_3 \to \frac{1}{2}T_3$, $Y \to -Y$). The particular choice of signs $p = p' = q' = -q = -1$ yields

$$Q = \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \tilde{T}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{17}$$

The first surprising result is the fact that only one value of $\tan^2 \theta_W$ is allowed, which means only one coupling constant (the electric charge, essentially, i.e., $e \equiv \tilde{r}' = \sqrt{2}r \equiv g/\sqrt{2}$), even though the gauge group $[U(2)]$ is not a simple group. According to general settings (Itzykson and Zuber, 1985), however, the theory must contain a coupling constant for each simple or Abelian term in the Lie algebra decomposition. An immediate conclusion is that the assignment of constants should be done according to factors in the direct product decomposition of the group, rather than the algebra.

The second result is the particular structure of the neutral weak current derived from the expression of $\tilde{T}_3$ above, according to which the gauge field Z^0 interacts with the (left-handed) neutrino and the right-handed electron only; i.e., the neutral weak current is pure V−A for the neutrino and pure V+A for the electron.

Last, but not least, is the striking value of $e^2/g^2 = 1/2$, far from the experimental value of $1 - M_W^2/M_Z^2 \approx 0.23$ (Aguilar-Benitez *et al.*, 1994). In the light of this result, only the hope remains that our theoretical characterization of θ_W really corresponds to that state of the universe in which the electroweak interaction was not yet "spontaneously broken," i.e., the masses of the vector bosons are zero and therefore the quantity $1 - M_W^2/M_Z^2$ makes no physical sense. In any case, our results provide strong support for the idea that the magnitude $1 - M_W^2/M_Z^2$ cannot be directly identified with e^2/g^2.

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